Time-Space Trade-Offs for Longest Common Extensions

Philip Bille[1], Inge Li Gørtz[1], Benjamin Sach[2], and Hjalte Wedel Vildhøj[1]

[1]Technical University of Denmark, DTU Informatics, {phbi,ilg,hwvi}@imm.dtu.dk
[2]University of Warwick, Department of Computer Science, sach@dcs.warwick.ac.uk

CPM 2012, Helsinki
July 4, 2012

The Longest Common Extension Problem

Definition

Problem: Preprocess a string T of length n to support LCE queries:

- $\text{LCE}(i,j) =$ The length of the longest common prefix of the suffixes starting at position i and j in T.

Example

$$T = \overset{1}{\texttt{b}}\ \overset{2}{\texttt{a}}\ \overset{3}{\texttt{n}}\ \overset{4}{\texttt{a}}\ \overset{5}{\texttt{n}}\ \overset{6}{\texttt{a}}\ \overset{7}{\texttt{s}} \qquad \text{LCE}(2,4) = \texttt{?}$$

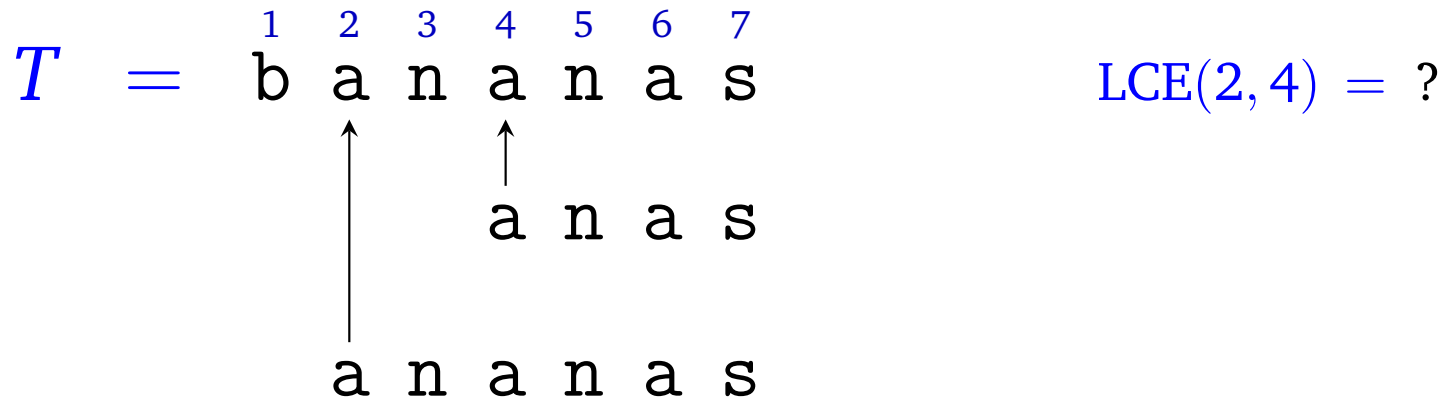
The Longest Common Extension Problem

Definition

Problem: Preprocess a string T of length n to support LCE queries:

- $\text{LCE}(i,j) =$ The length of the longest common prefix of the suffixes starting at position i and j in T.

Example



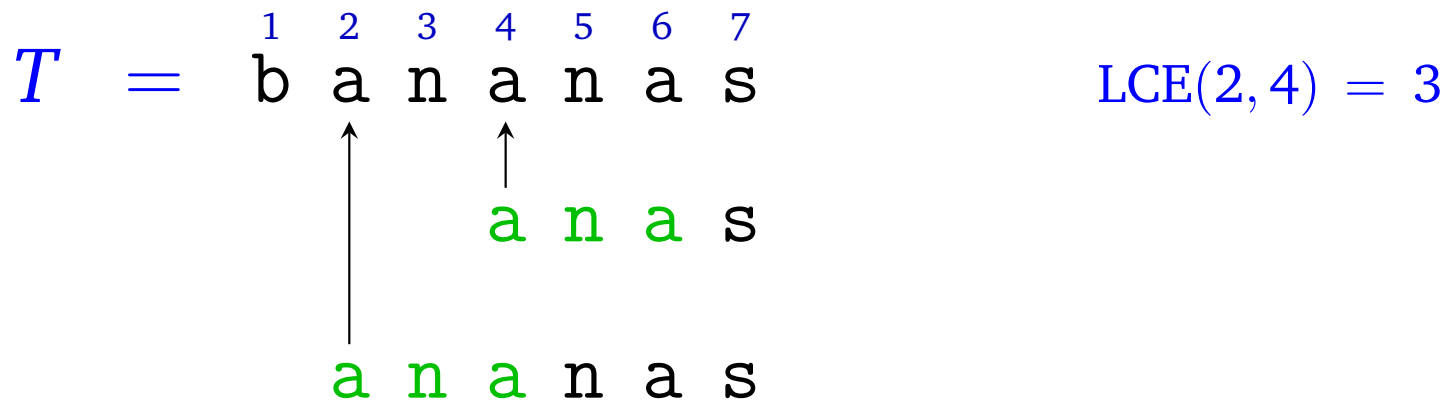
The Longest Common Extension Problem

Definition

Problem: Preprocess a string T of length n to support LCE queries:

- $\text{LCE}(i,j) =$ The length of the longest common prefix of the suffixes starting at position i and j in T.

Example



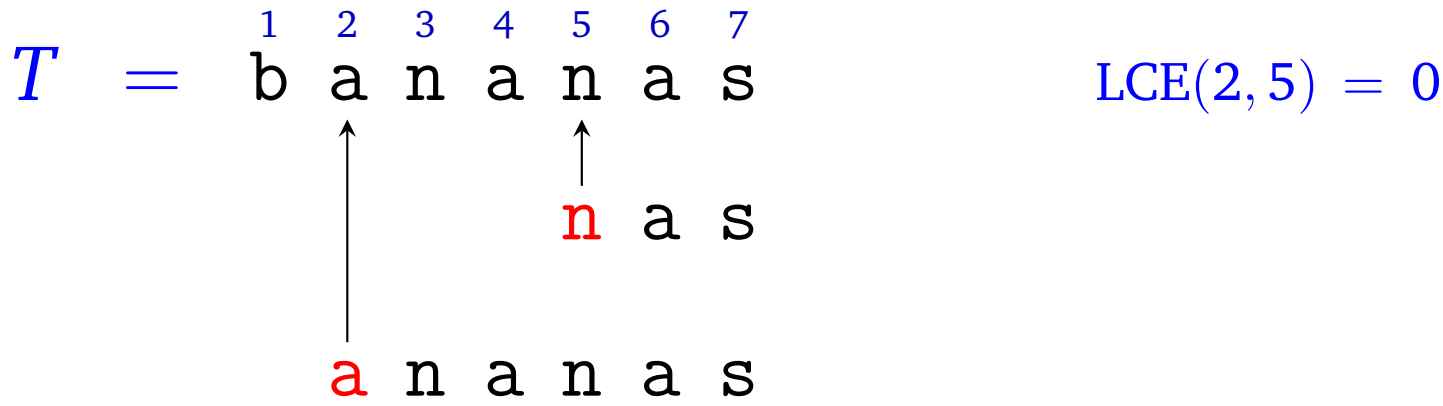
The Longest Common Extension Problem

Definition

Problem: Preprocess a string T of length n to support LCE queries:

- $\text{LCE}(i,j) =$ The length of the longest common prefix of the suffixes starting at position i and j in T.

Example



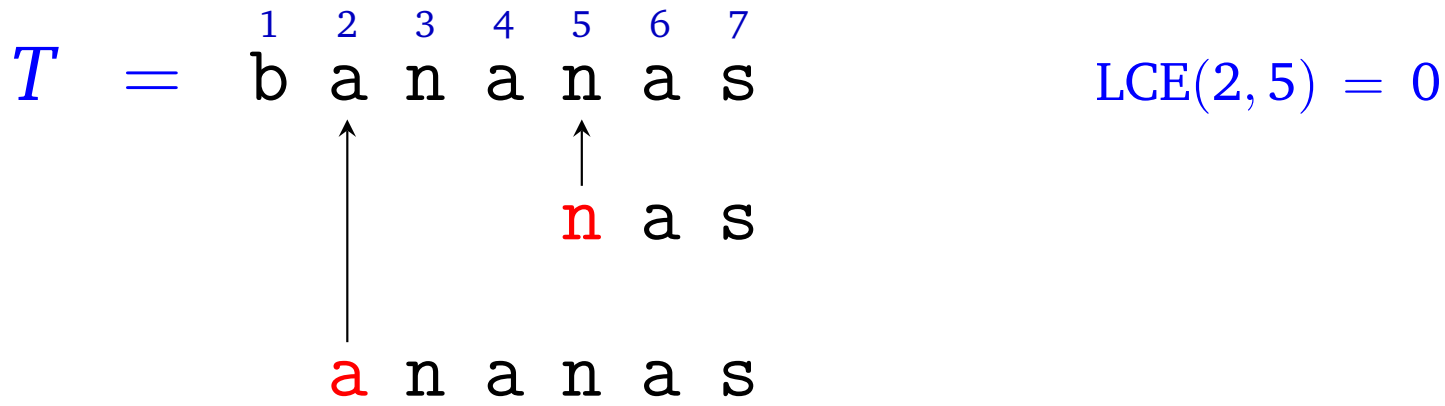
The Longest Common Extension Problem

Definition

Problem: Preprocess a string T of length n to support LCE queries:

- $\text{LCE}(i,j) =$ The length of the longest common prefix of the suffixes starting at position i and j in T.

Example



- We assume that the input is given in read-only memory and is not included in the space complexity.

Two Simple Solutions

#1: Store nothing

$$T = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \texttt{b} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{s} \\ & \uparrow & & \uparrow & & & \\ & i & & j & & & \end{array} \qquad \mathrm{LCE}(i,j) =$$

Two Simple Solutions

#1: Store nothing

$$T = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \texttt{b} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{s} \\ & \uparrow & & \uparrow & & & \\ & i & & j & & & \end{array} \qquad \mathrm{LCE}(i,j) = 1$$

Two Simple Solutions

#1: Store nothing

$$T = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \texttt{b} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{s} \\ & \uparrow & & \uparrow & & & \\ & i & & j & & & \end{array} \qquad \mathrm{LCE}(i,j) = 2$$

Two Simple Solutions

#1: Store nothing

$$T = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \texttt{b} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{s} \\ & \uparrow & & \uparrow & & & \\ & i & & j & & & \end{array} \qquad \mathrm{LCE}(i,j) = 3$$

Two Simple Solutions

#1: Store nothing

$$T = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \texttt{b} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{s} \\ & \uparrow & & \uparrow & & & \\ & i & & j & & & \end{array}$$

$\text{LCE}(i,j) = 3$

Time:	$O(n)$
Space:	$O(1)$

Two Simple Solutions

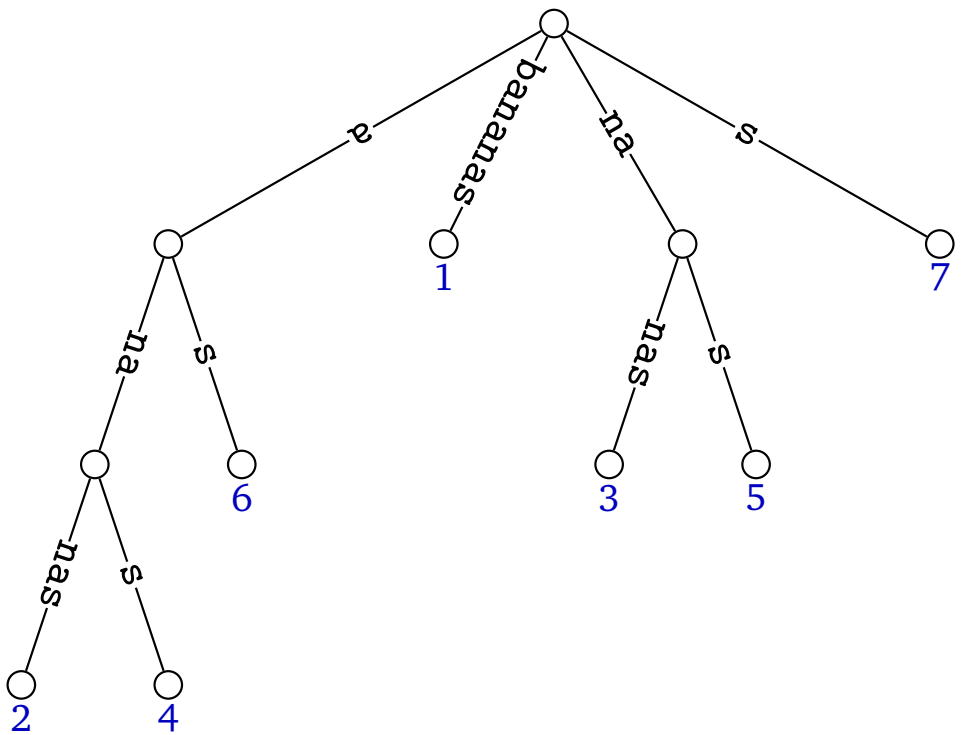
#1: Store nothing

$$T = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \texttt{b} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{s} \\ & \uparrow & & \uparrow & & & \\ & i & & j & & & \end{array}$$

$\text{LCE}(i,j) = 3$

Time:	$O(n)$
Space:	$O(1)$

#2: Store the suffix tree



Two Simple Solutions

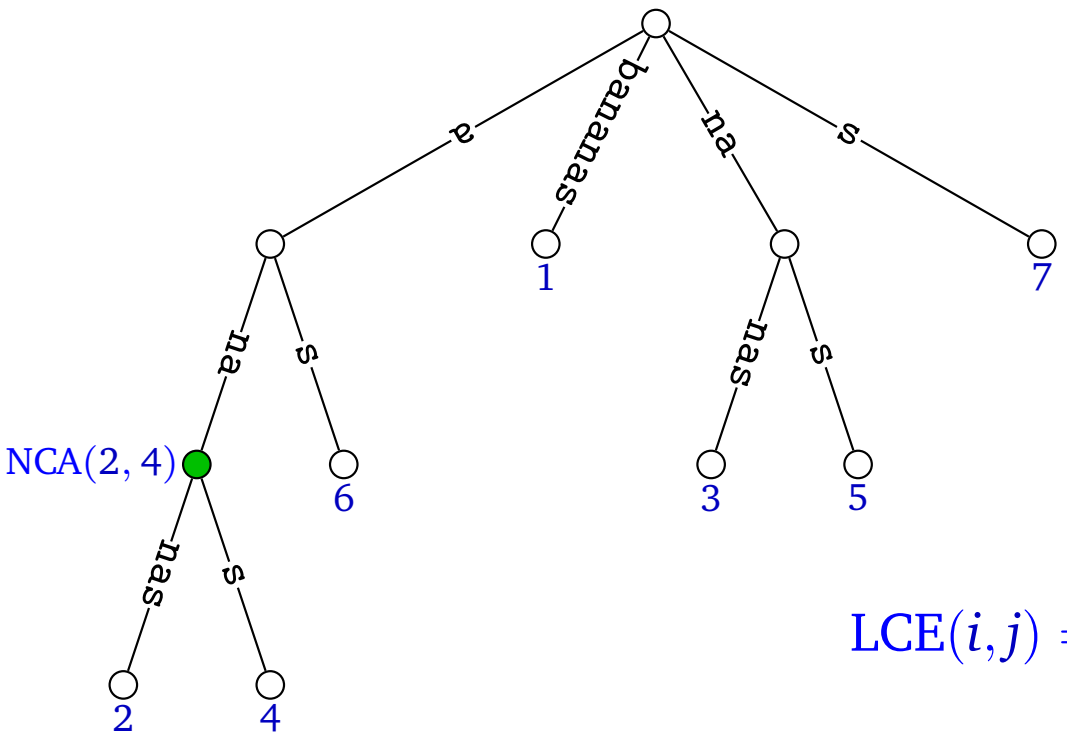
#1: Store nothing

$$T = \begin{array}{ccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ & \mathtt{b} & \mathtt{a} & \mathtt{n} & \mathtt{a} & \mathtt{n} & \mathtt{a} & \mathtt{s} & \\ & & \uparrow & & \uparrow & & & & \\ & & i & & j & & & & \end{array}$$

$\mathrm{LCE}(i,j) = 3$

Time:	$O(n)$
Space:	$O(1)$

#2: Store the suffix tree



$\mathrm{LCE}(i,j) = |\mathrm{NCA}(i,j)| = 3$

Two Simple Solutions

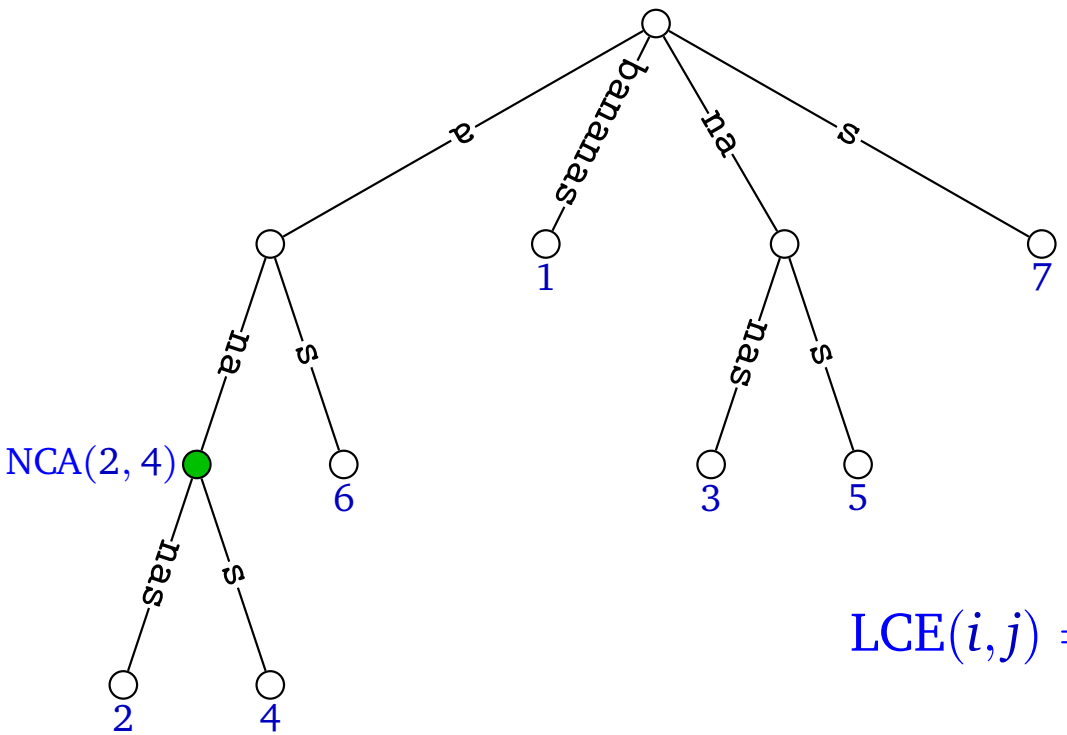
#1: Store nothing

$$T = \begin{array}{ccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ & \mathtt{b} & \mathtt{a} & \mathtt{n} & \mathtt{a} & \mathtt{n} & \mathtt{a} & \mathtt{s} & \\ & & \uparrow & & \uparrow & & & & \\ & & i & & j & & & & \end{array}$$

$\mathrm{LCE}(i,j) = 3$

Time:	$O(n)$
Space:	$O(1)$

#2: Store the suffix tree



Time:	$O(1)$
Space:	$O(n)$

$\mathrm{LCE}(i,j) = |\mathrm{NCA}(i,j)| = 3$

Two Simple Solutions

#1: Store nothing

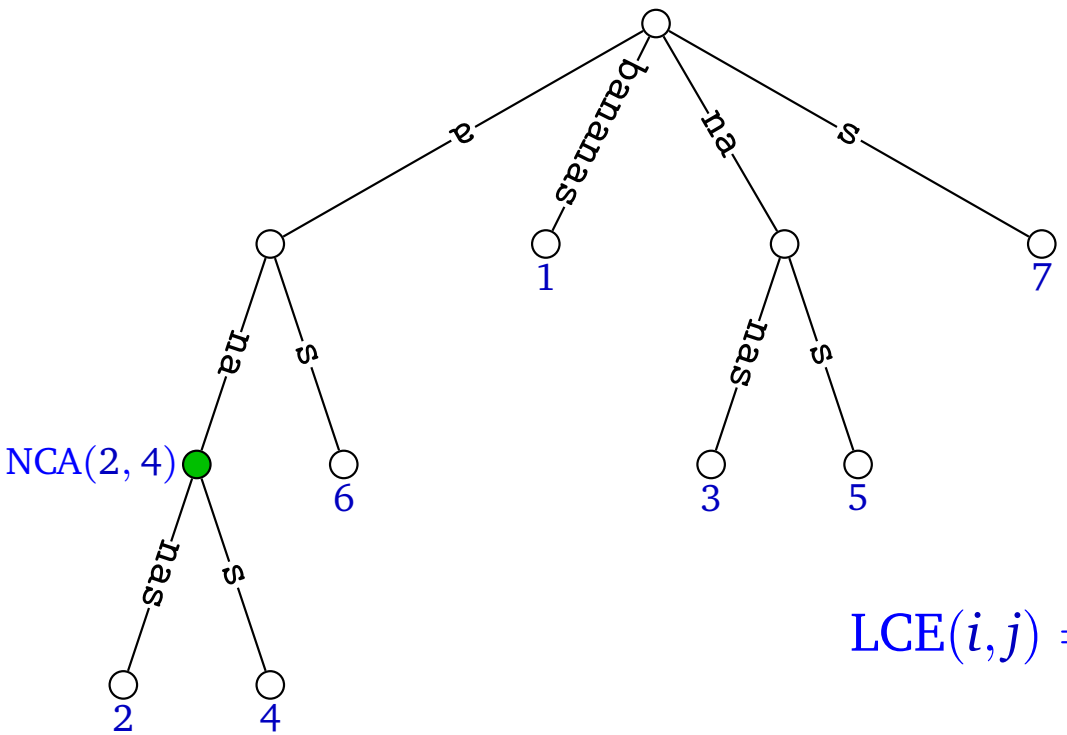
$$T = \underset{1}{\texttt{b}}\ \underset{2}{\texttt{a}}\ \underset{3}{\texttt{n}}\ \underset{4}{\texttt{a}}\ \underset{5}{\texttt{n}}\ \underset{6}{\texttt{a}}\ \underset{7}{\texttt{s}}$$

i j

$\mathrm{LCE}(i,j) = 3$

Time: $O(n)$
Space: $O(1)$

#2: Store the suffix tree



Trade-off?

Time: $O(1)$
Space: $O(n)$

$\mathrm{LCE}(i,j) = |\mathrm{NCA}(i,j)| = 3$

Our Results

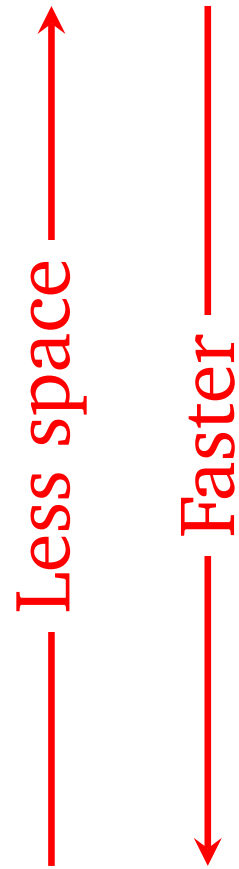
Store nothing

Time: $O(n)$
Space: $O(1)$

Trade-off?

Time: $O(1)$
Space: $O(n)$

Store suffix tree



Our Results

Trade-off parameter τ, $1 \leq \tau \leq n$

Store nothing

Time: $O(n)$
Space: $O(1)$

Trade-off?

Time: $O(1)$
Space: $O(n)$

Store suffix tree

Randomized

Time: $O\left(\tau \log\left(\frac{\mathrm{LCE}(i,j)}{\tau}\right)\right)$
Space: $O\left(\frac{n}{\tau}\right)$

Time: $O(\tau)$
Space: $O\left(\frac{n}{\sqrt{\tau}}\right)$

Deterministic

Less space

Faster

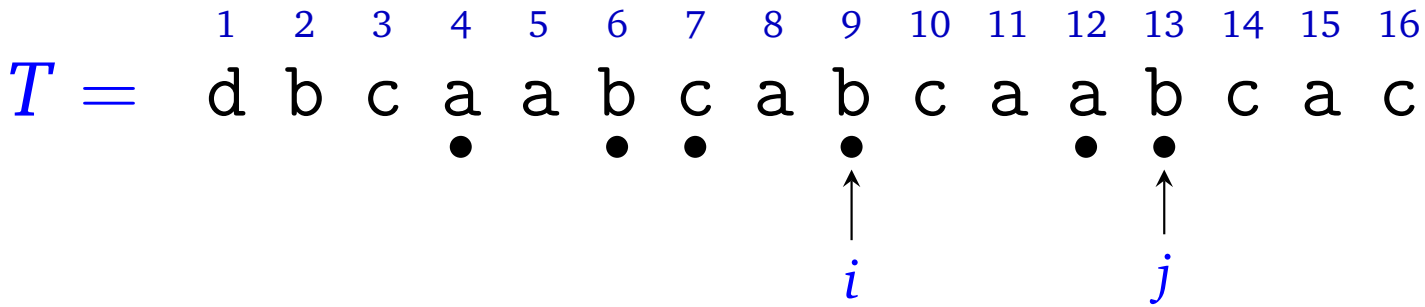
A Deterministic Solution

Idea: Store a subset of the n suffixes in a compacted trie.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$T =$	d	b	c	a	a	b	c	a	b	c	a	a	b	c	a	c
				•		•	•		•			•	•			

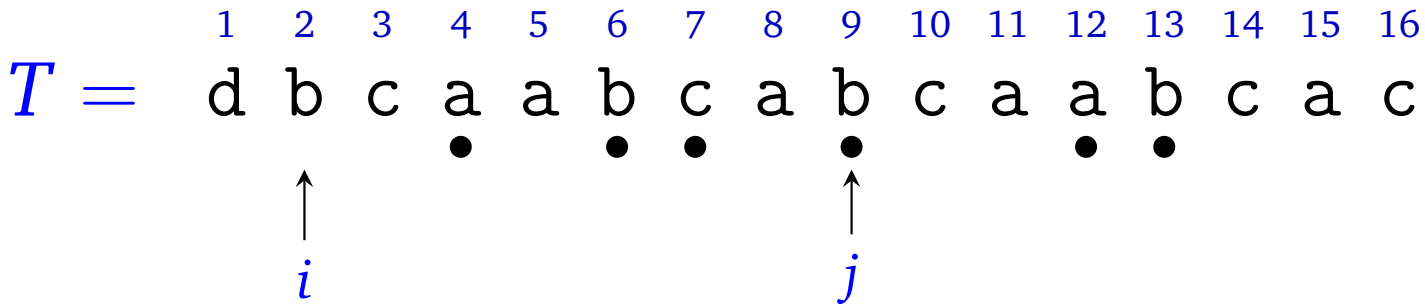
A Deterministic Solution

Idea: Store a subset of the n suffixes in a compacted trie.



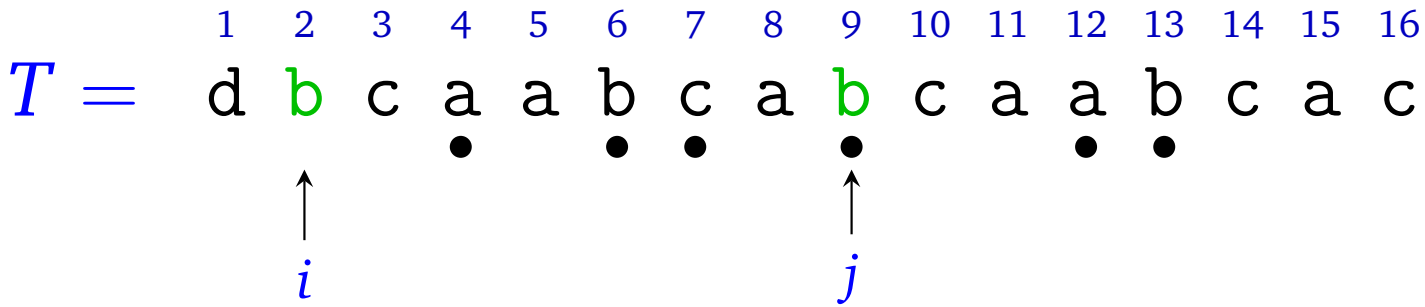
A Deterministic Solution

Idea: Store a subset of the n suffixes in a compacted trie.



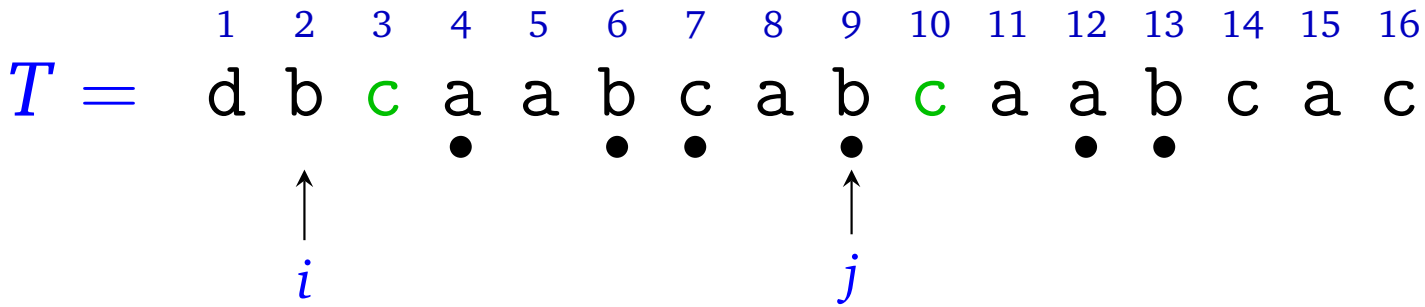
A Deterministic Solution

Idea: Store a subset of the n suffixes in a compacted trie.



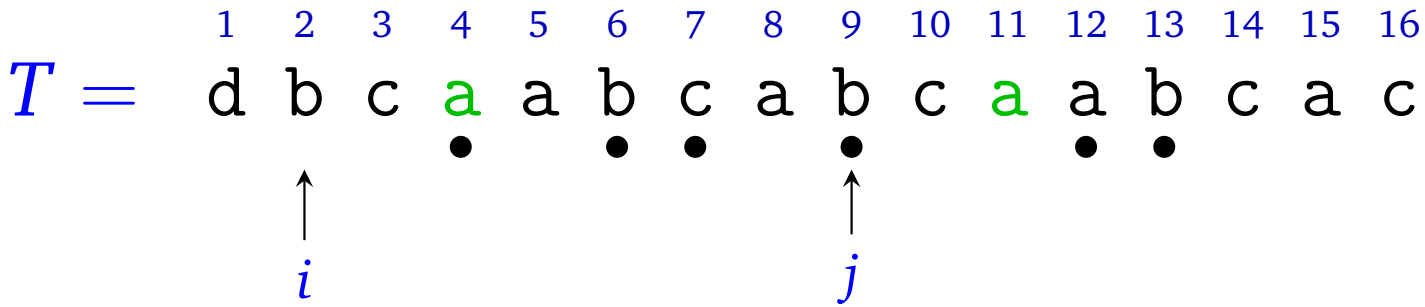
A Deterministic Solution

Idea: Store a subset of the n suffixes in a compacted trie.



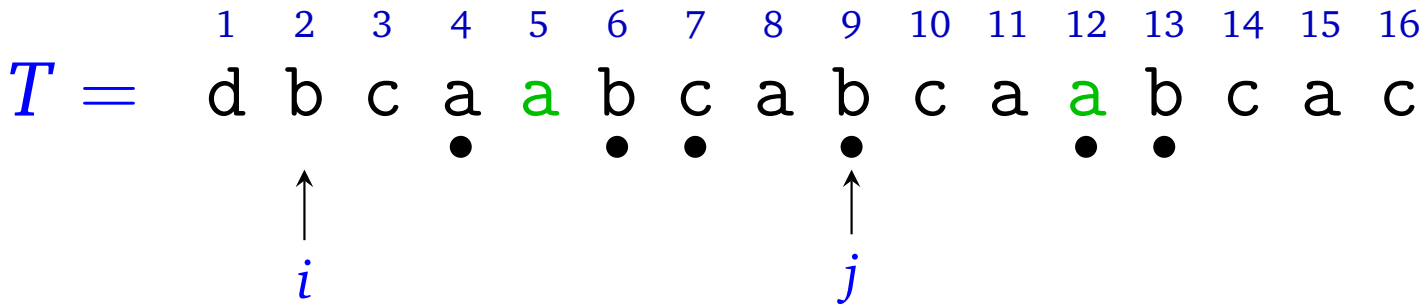
A Deterministic Solution

Idea: Store a subset of the n suffixes in a compacted trie.



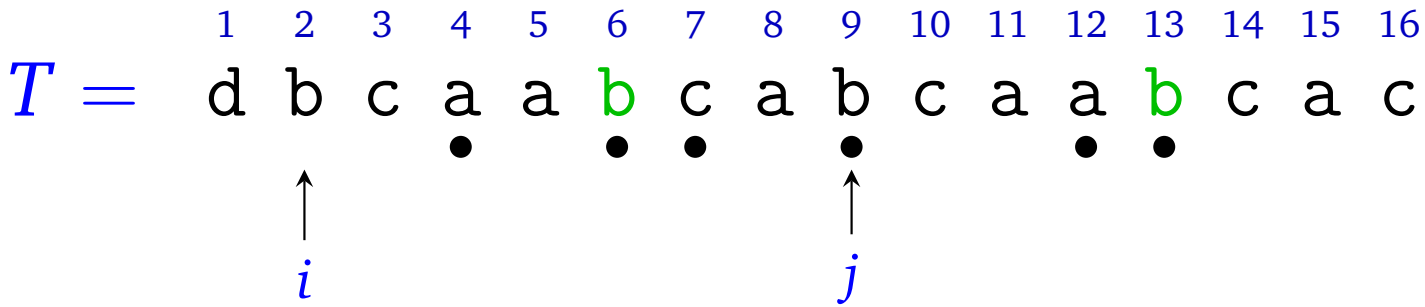
A Deterministic Solution

Idea: Store a subset of the n suffixes in a compacted trie.



A Deterministic Solution

Idea: Store a subset of the n suffixes in a compacted trie.



A Deterministic Solution

Idea: Store a subset of the n suffixes in a compacted trie.

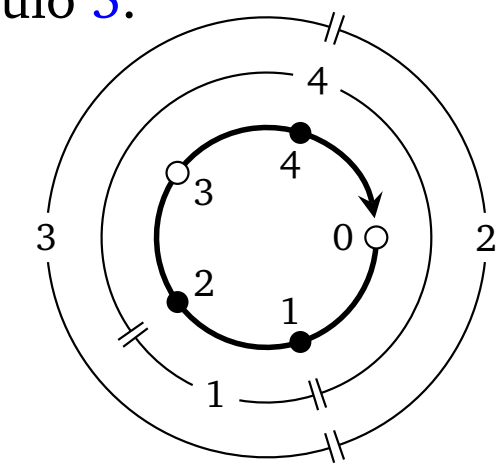
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$T =$	d	b	c	a	a	b	c	a	b	c	a	a	b	c	a	c

Difference Covers

A *difference cover modulo* τ is a set of integers $D \subseteq \{0, 1, \ldots, \tau - 1\}$ such that for any distance $d \in \{0, 1, \ldots, \tau - 1\}$, D contains two elements separated by distance d modulo τ.

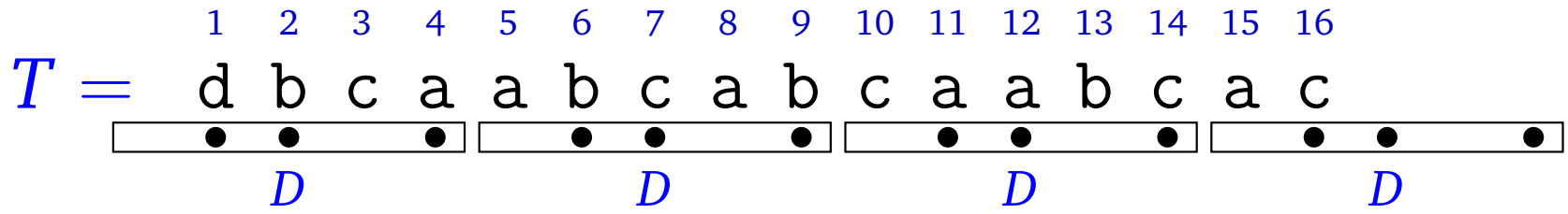
Ex: The set $D = \{1, 2, 4\}$ is a difference cover modulo 5.

d	0	1	2	3	4
i, j	1, 1	2, 1	1, 4	4, 1	1, 2



A Deterministic Solution

Idea: Store a subset of the n suffixes in a compacted trie.

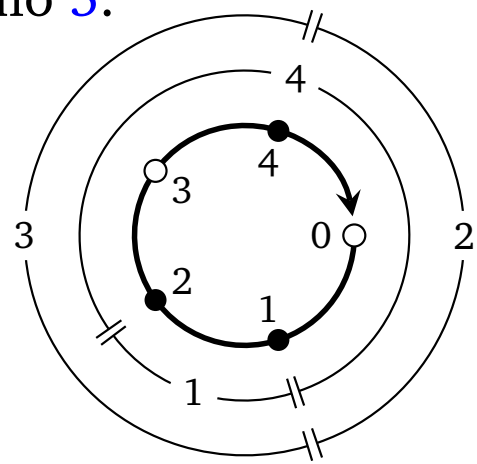


Difference Covers

A *difference cover modulo* τ is a set of integers $D \subseteq \{0, 1, \ldots, \tau - 1\}$ such that for any distance $d \in \{0, 1, \ldots, \tau - 1\}$, D contains two elements separated by distance d modulo τ.

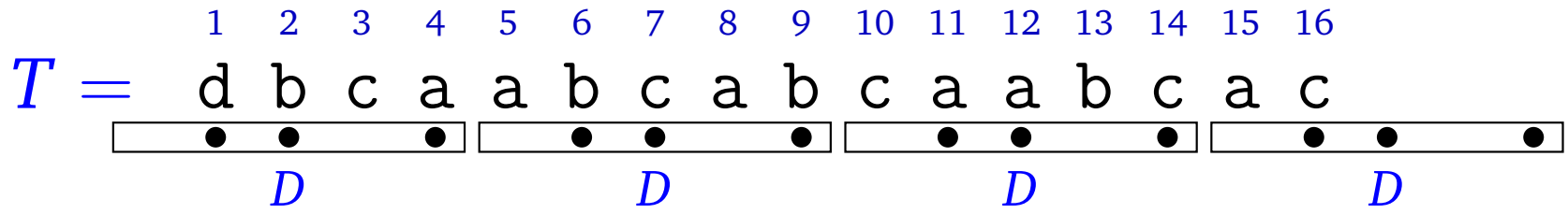
Ex: The set $D = \{1, 2, 4\}$ is a difference cover modulo 5.

d	0	1	2	3	4
i, j	1, 1	2, 1	1, 4	4, 1	1, 2



A Deterministic Solution

Idea: Store a subset of the n suffixes in a compacted trie.



Lemma (Colbourn and Ling[1])

For any τ, a difference cover modulo τ of size at most $\sqrt{1.5\tau} + 6$ can be computed in $O(\sqrt{\tau})$ time.

Analysis

Time: $O(\tau)$

Space: $O(\#\text{stored suffixes}) = O\left(\frac{n}{\tau}|D|\right) = O\left(\frac{n}{\sqrt{\tau}}\right)$

[1] C. J. Colbourn and A. C. Ling. Quorums from difference covers. Inf. Process. Lett. 75(1-2):9–12, 2000

A Randomized Solution (Monte Carlo)

Rabin-Karp Fingerprints

Let p be a sufficiently large prime and choose $b \in \mathbb{Z}_p$ uniformly at random.

$$\phi(S) = \sum_{k=1}^{|S|} S[k]b^k \bmod p\,.$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$T =$	d	b	c	a	a	b	c	a	b	c	a	a	b	c	a	c

A Randomized Solution (Monte Carlo)

Rabin-Karp Fingerprints

Let p be a sufficiently large prime and choose $b \in \mathbb{Z}_p$ uniformly at random.

$$\phi(S) = \sum_{k=1}^{|S|} S[k]b^k \bmod p\,.$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$T =$	d	b	c	a	a	b	c	a	b	c	a	a	b	c	a	c
$=$	3	1	2	0	0	1	2	0	1	2	0	0	1	2	0	2

$$\phi(T[2\ldots7]) = 1b^1 + 2b^2 + 0b^3 + 0b^4 + 1b^5 + 2b^6 \bmod p$$

A Randomized Solution (Monte Carlo)

Rabin-Karp Fingerprints

Let p be a sufficiently large prime and choose $b \in \mathbb{Z}_p$ uniformly at random.

$$\phi(S) = \sum_{k=1}^{|S|} S[k]b^k \bmod p\,.$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$T =$	d	b	c	a	a	b	c	a	b	c	a	a	b	c	a	c
$=$	3	1	2	0	0	1	2	0	1	2	0	0	1	2	0	2

$$\phi(T[2\ldots7]) = 1b^1 + 2b^2 + 0b^3 + 0b^4 + 1b^5 + 2b^6 \bmod p$$

Crucial property: With high probability ϕ is collision-free on substrings of T, i.e., $\phi(S_1) = \phi(S_2)$ iff $S_1 = S_2$.

A Randomized Solution (Monte Carlo)

Rabin-Karp Fingerprints

Let p be a sufficiently large prime and choose $b \in \mathbb{Z}_p$ uniformly at random.

$$\phi(S) = \sum_{k=1}^{|S|} S[k]b^k \bmod p\,.$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$T =$	d	b	c	a	a	b	c	a	b	c	a	a	b	c	a	c
$=$	3	1	2	0	0	1	2	0	1	2	0	0	1	2	0	2

$$\phi(T[2\ldots7]) = 1b^1 + 2b^2 + 0b^3 + 0b^4 + 1b^5 + 2b^6 \bmod p$$

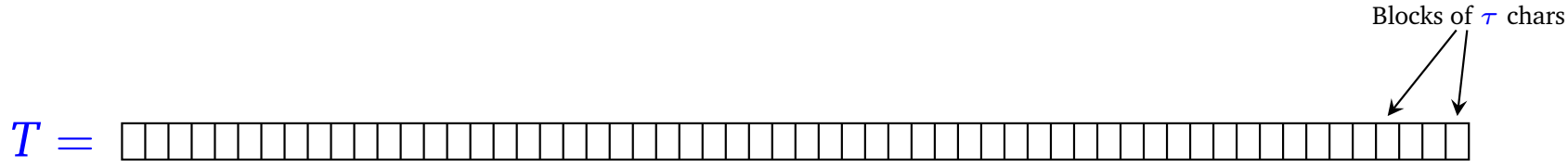
Crucial property: With high probability ϕ is collision-free on substrings of T, i.e., $\phi(S_1) = \phi(S_2)$ iff $S_1 = S_2$.

Also important: $\phi(T[i\ldots j+1])$ can be computed from $\phi(T[i\ldots j])$ in $O(1)$ time.

A Randomized Solution (Monte Carlo)

How to answer a query

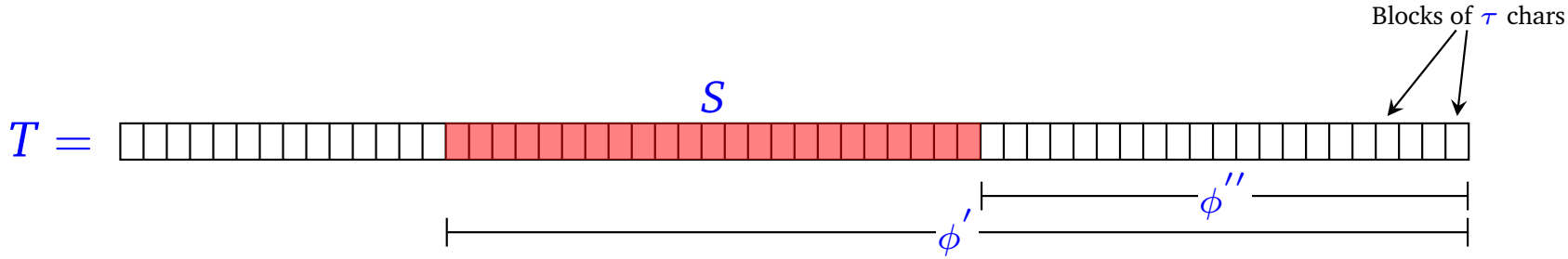
Idea: Store fingerprints of suffixes starting at every τ'th position in T.



A Randomized Solution (Monte Carlo)

How to answer a query

Idea: Store fingerprints of suffixes starting at every τ'th position in T.

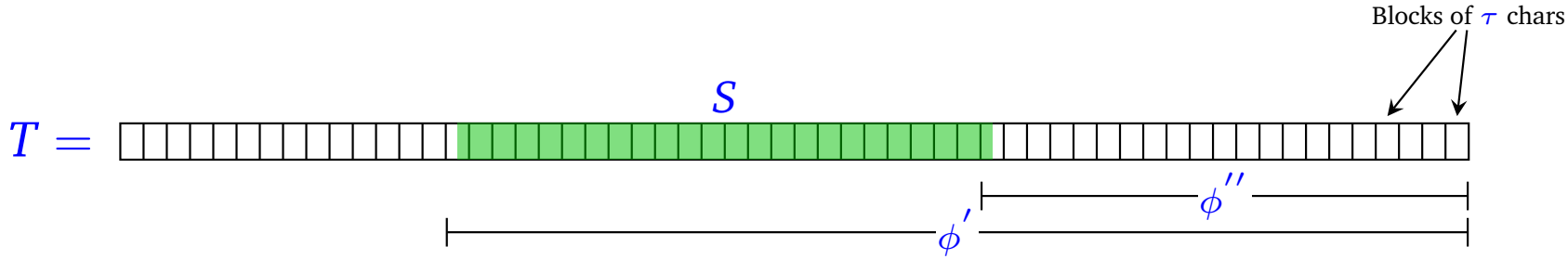


Observation: If S is block aligned we can compute $\phi(S)$ in $O(1)$ time. Otherwise, the time needed is $O(\tau)$.

A Randomized Solution (Monte Carlo)

How to answer a query

Idea: Store fingerprints of suffixes starting at every τ'th position in T.

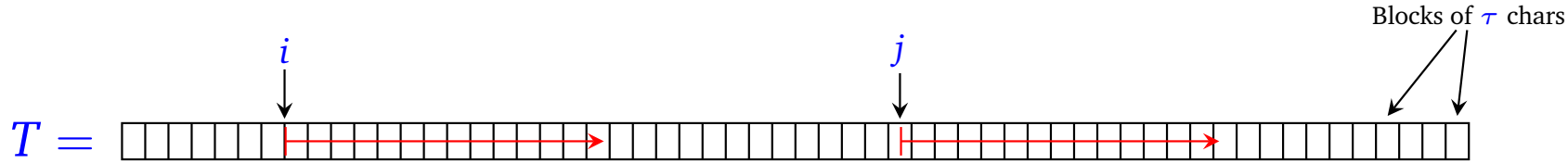


Observation: If S is block aligned we can compute $\phi(S)$ in $O(1)$ time. Otherwise, the time needed is $O(\tau)$.

A Randomized Solution (Monte Carlo)

How to answer a query

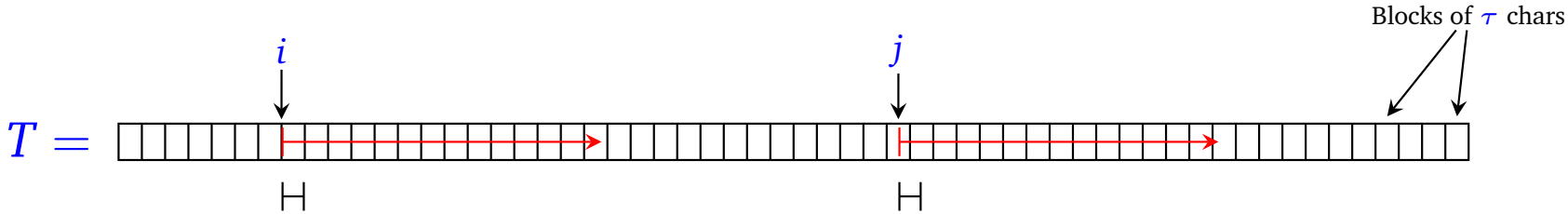
Idea: Store fingerprints of suffixes starting at every τ'th position in T.



A Randomized Solution (Monte Carlo)

How to answer a query

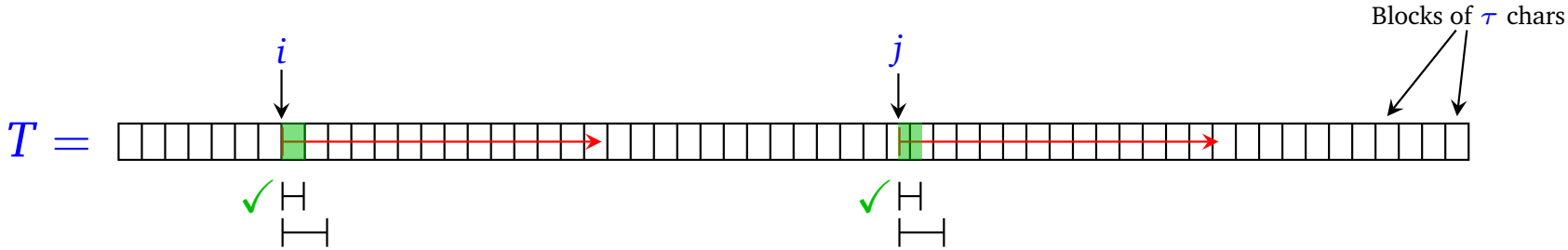
Idea: Store fingerprints of suffixes starting at every τ'th position in T.



A Randomized Solution (Monte Carlo)

How to answer a query

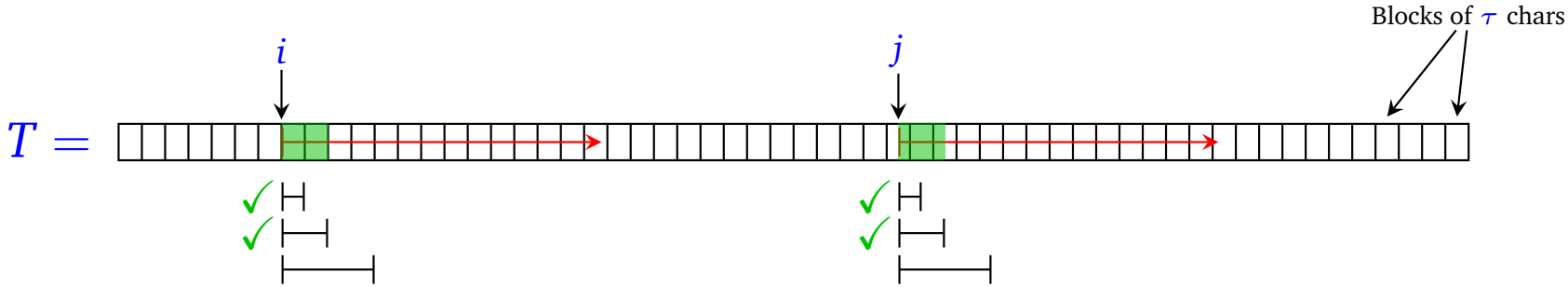
Idea: Store fingerprints of suffixes starting at every τ'th position in T.



A Randomized Solution (Monte Carlo)

How to answer a query

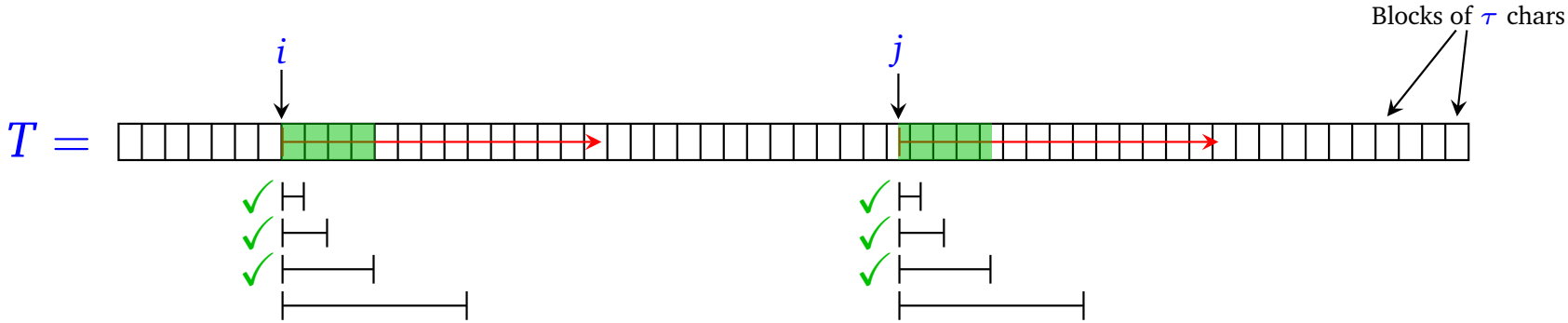
Idea: Store fingerprints of suffixes starting at every τ'th position in T.



A Randomized Solution (Monte Carlo)

How to answer a query

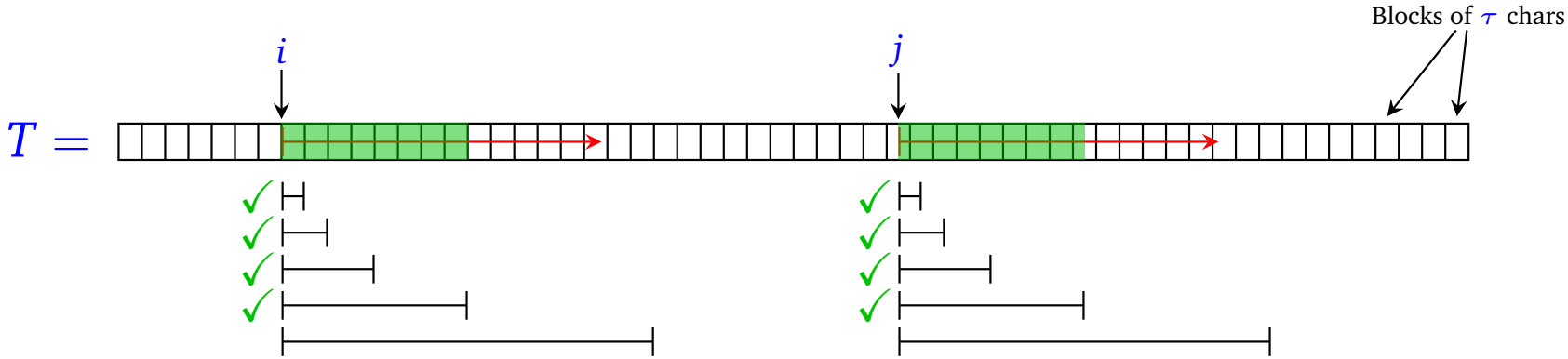
Idea: Store fingerprints of suffixes starting at every τ'th position in T.



A Randomized Solution (Monte Carlo)

How to answer a query

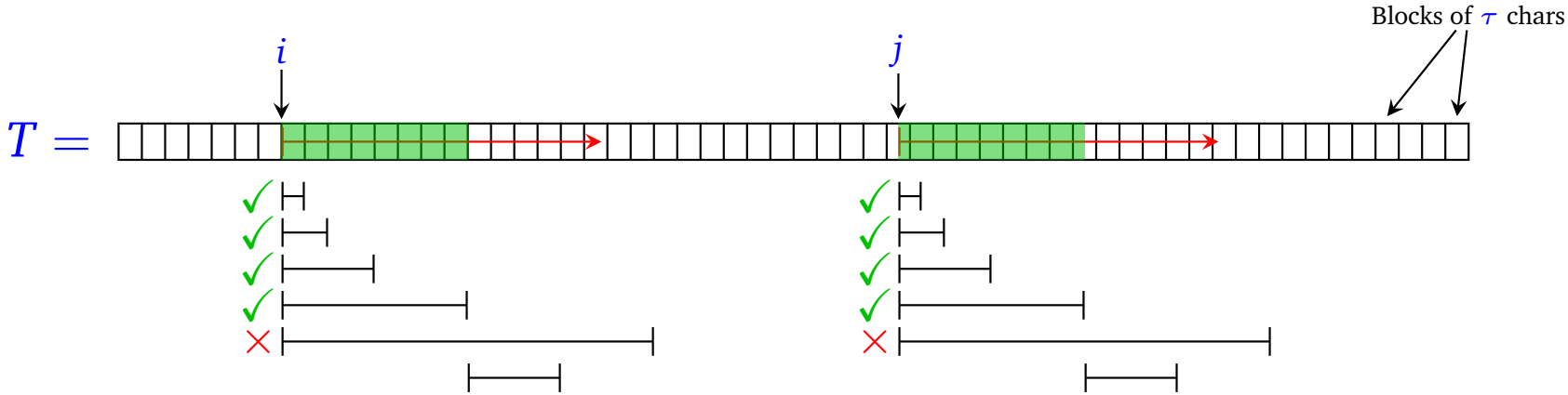
Idea: Store fingerprints of suffixes starting at every τ'th position in T.



A Randomized Solution (Monte Carlo)

How to answer a query

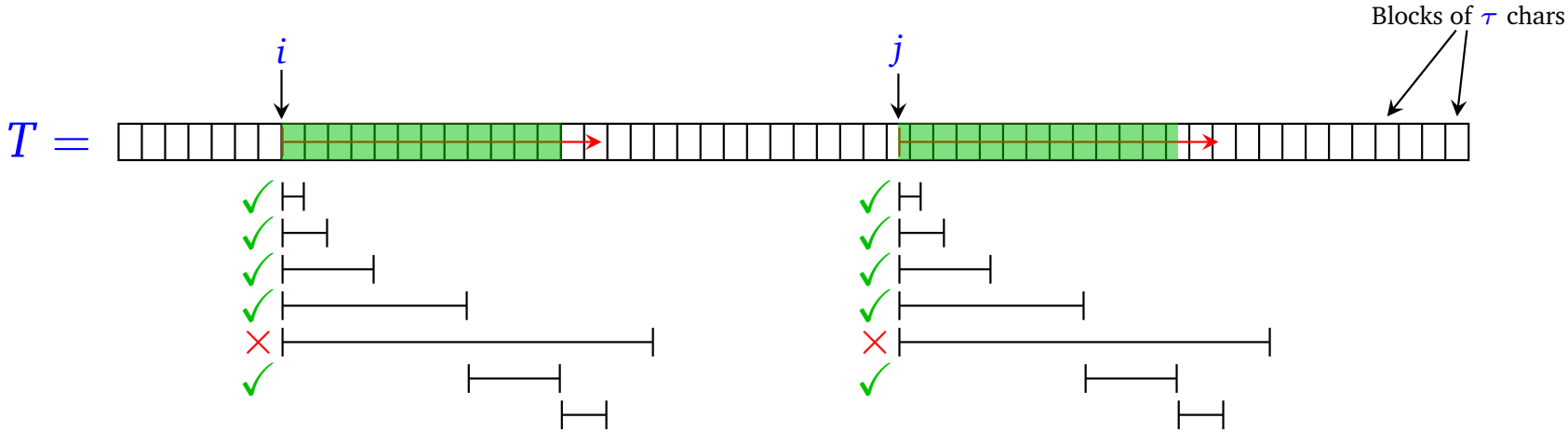
Idea: Store fingerprints of suffixes starting at every τ'th position in T.



A Randomized Solution (Monte Carlo)

How to answer a query

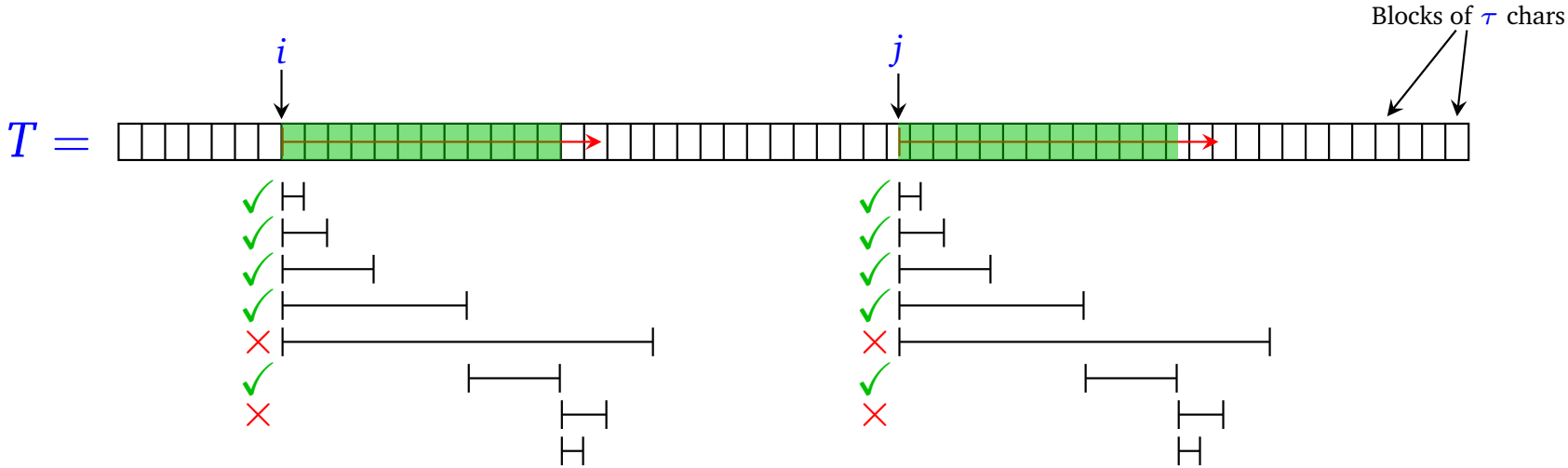
Idea: Store fingerprints of suffixes starting at every τ'th position in T.



A Randomized Solution (Monte Carlo)

How to answer a query

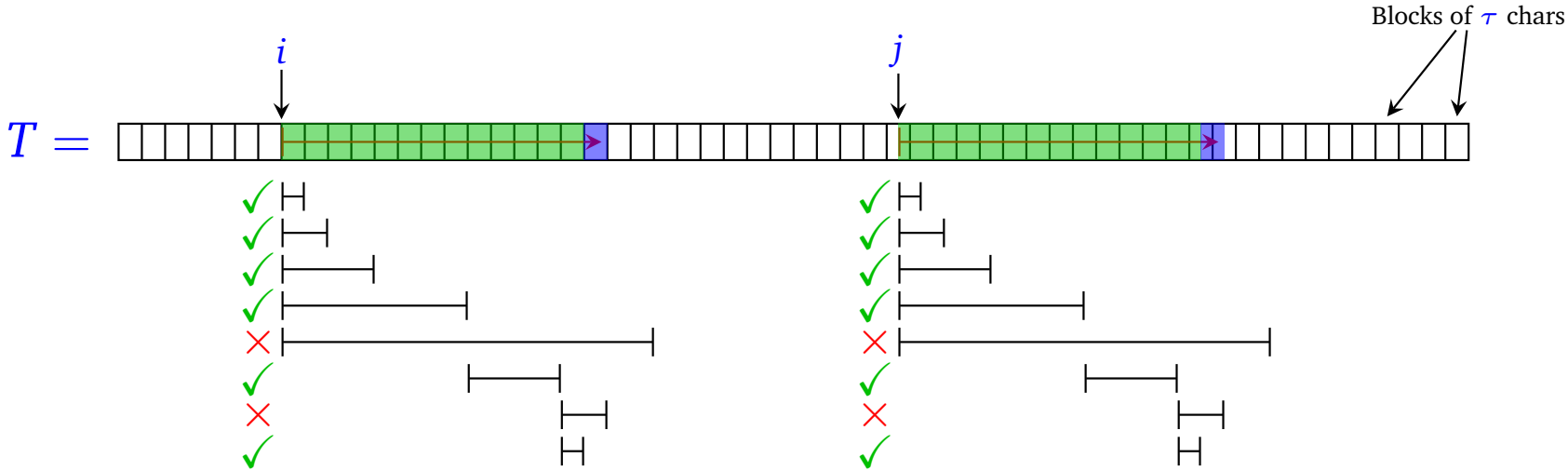
Idea: Store fingerprints of suffixes starting at every τ'th position in T.



A Randomized Solution (Monte Carlo)

How to answer a query

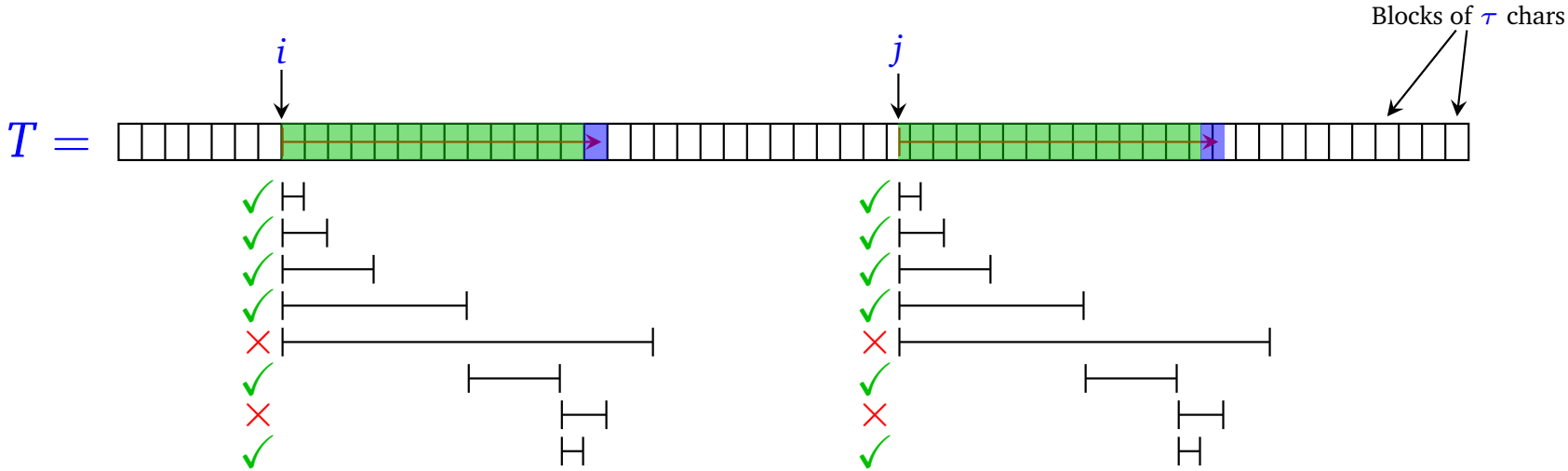
Idea: Store fingerprints of suffixes starting at every τ'th position in T.



A Randomized Solution (Monte Carlo)

How to answer a query

Idea: Store fingerprints of suffixes starting at every τ'th position in T.



Analysis

Time: Only $O(\log(\frac{\text{LCE}}{\tau}))$ fingerprint comparisons each taking time $O(\tau)$. Hence query time $O\left(\tau \log\left(\frac{\text{LCE}}{\tau}\right)\right)$.

Space: $O\left(\frac{n}{\tau}\right)$.

A Randomized Solution (Las Vegas)

Question: Can we verify that ϕ is collision free during preprocessing?

A Randomized Solution (Las Vegas)

Question: Can we verify that ϕ is collision free during preprocessing?

Challenge: Doing this quickly while using $O(\frac{n}{\tau})$ space.

A Randomized Solution (Las Vegas)

Question: Can we verify that ϕ is collision free during preprocessing?

Challenge: Doing this quickly while using $O(\frac{n}{\tau})$ space.

Observation: Whenever we compare two fingerprints, we can ensure that one of them is of the form $T[j\tau \ldots j\tau + \tau \cdot 2^{\ell} - 1]$ for some ℓ, j

A Randomized Solution (Las Vegas)

Question: Can we verify that ϕ is collision free during preprocessing?

Challenge: Doing this quickly while using $O(\frac{n}{\tau})$ space.

Observation: Whenever we compare two fingerprints, we can ensure that one of them is of the form $T[j\tau \ldots j\tau + \tau \cdot 2^{\ell} - 1]$ for some ℓ, j

. . . this cuts down the number of fingerprints we need to check!

A Randomized Solution (Las Vegas)

Question: Can we verify that ϕ is collision free during preprocessing?

Challenge: Doing this quickly while using $O(\frac{n}{\tau})$ space.

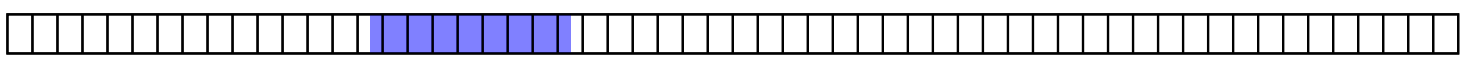
Observation: Whenever we compare two fingerprints, we can ensure that one of them is of the form $T[j\tau \ldots j\tau + \tau \cdot 2^\ell - 1]$ for some ℓ, j

. . . this cuts down the number of fingerprints we need to check!

General idea: For each $\ell \geq 0$ in increasing order, check that for all i, j,

$$\phi(T[i \ldots i + \tau \cdot 2^\ell - 1]) = \phi(T[j\tau \ldots j\tau + \tau \cdot 2^\ell - 1])$$

$$\text{iff} \quad T[i \ldots i + \tau \cdot 2^\ell - 1] = T[j\tau \ldots j\tau + \tau \cdot 2^\ell - 1]$$

$T =$ 

A Randomized Solution (Las Vegas)

Question: Can we verify that ϕ is collision free during preprocessing?

Challenge: Doing this quickly while using $O(\frac{n}{\tau})$ space.

Observation: Whenever we compare two fingerprints, we can ensure that one of them is of the form $T[j\tau \ldots j\tau + \tau \cdot 2^\ell - 1]$ for some ℓ, j

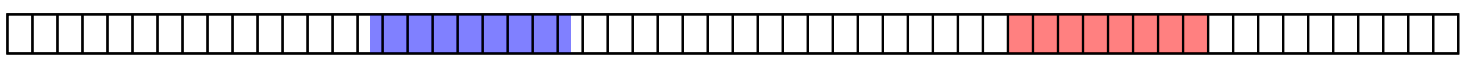
. . . this cuts down the number of fingerprints we need to check!

General idea: For each $\ell \geq 0$ in increasing order, check that for all i, j,

$$\phi(T[i \ldots i + \tau \cdot 2^\ell - 1]) = \phi(T[j\tau \ldots j\tau + \tau \cdot 2^\ell - 1])$$

$$\text{iff} \quad T[i \ldots i + \tau \cdot 2^\ell - 1] = T[j\tau \ldots j\tau + \tau \cdot 2^\ell - 1]$$

$$\phi(T[i \ldots i + \tau \cdot 2^\ell - 1]) = \phi(T[j\tau \ldots j\tau + \tau \cdot 2^\ell - 1])$$

$T =$ 

A Randomized Solution (Las Vegas)

Question: Can we verify that ϕ is collision free during preprocessing?

Challenge: Doing this quickly while using $O(\frac{n}{\tau})$ space.

Observation: Whenever we compare two fingerprints, we can ensure that one of them is of the form $T[j\tau \ldots j\tau + \tau \cdot 2^\ell - 1]$ for some ℓ, j

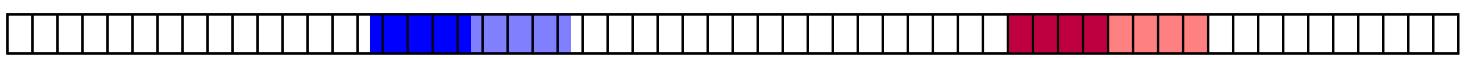
. . . this cuts down the number of fingerprints we need to check!

General idea: For each $\ell \geq 0$ in increasing order, check that for all i, j,

$$\phi(T[i \ldots i + \tau \cdot 2^\ell - 1]) = \phi(T[j\tau \ldots j\tau + \tau \cdot 2^\ell - 1])$$

$$\text{iff} \quad T[i \ldots i + \tau \cdot 2^\ell - 1] = T[j\tau \ldots j\tau + \tau \cdot 2^\ell - 1]$$

$$T = \quad \phi(T[i \ldots i + \tau \cdot 2^\ell - 1]) \quad = \phi(T[j\tau \ldots j\tau + \tau \cdot 2^\ell - 1])$$



$$\phi(T[i \ldots i + \tau \cdot 2^{\ell-1} - 1]) \stackrel{?}{=} \phi(T[j\tau \ldots j\tau + \tau \cdot 2^{\ell-1} - 1])$$

A Randomized Solution (Las Vegas)

Question: Can we verify that ϕ is collision free during preprocessing?

Challenge: Doing this quickly while using $O(\frac{n}{\tau})$ space.

Observation: Whenever we compare two fingerprints, we can ensure that one of them is of the form $T[j\tau \ldots j\tau + \tau \cdot 2^\ell - 1]$ for some ℓ, j

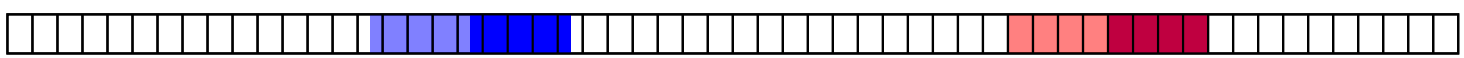
. . . this cuts down the number of fingerprints we need to check!

General idea: For each $\ell \geq 0$ in increasing order, check that for all i, j,

$$\phi(T[i \ldots i + \tau \cdot 2^\ell - 1]) = \phi(T[j\tau \ldots j\tau + \tau \cdot 2^\ell - 1])$$

$$\text{iff} \quad T[i \ldots i + \tau \cdot 2^\ell - 1] = T[j\tau \ldots j\tau + \tau \cdot 2^\ell - 1]$$

$$T = \quad \phi(T[i \ldots i + \tau \cdot 2^\ell - 1]) \quad = \phi(T[j\tau \ldots j\tau + \tau \cdot 2^\ell - 1])$$



$$\phi(T[i + \tau \cdot 2^{\ell-1} \ldots i + \tau \cdot 2^\ell - 1])$$

$$\stackrel{?}{=} \phi(T[j\tau + \tau \cdot 2^{\ell-1} \ldots j\tau + \tau \cdot 2^\ell - 1])$$

Conclusions

We gave three time-space trade-offs for LCE on a single string:

- A deterministic solution
  - $O(\tau)$ query time
  - $O(n/\sqrt{\tau})$ space (even during preprocessing)
  - $O(n^2/\sqrt{\tau})$ preprocessing time
- A Monte-Carlo solution
  - $O\left(\tau \log\left(\mathrm{LCE}(i,j)/\tau\right)\right)$ query time (correct with high prob.)
  - $O(n/\tau)$ space (even during preprocessing)
  - $O(n)$ preprocessing time.
- A Las-Vegas solution
  - $O\left(\tau \log\left(\mathrm{LCE}(i,j)/\tau\right)\right)$ query time (correct with certainty)
  - $O(n/\tau)$ space (even during preprocessing)
  - $O(n \log n)$ preprocessing time with high prob.

Conclusions

We gave three time-space trade-offs for LCE on two strings:

- A deterministic solution
  - $O(\tau)$ query time
  - $O(n/\tau + m/\sqrt{\tau})$ space (even during preprocessing)
  - $O(nm/\sqrt{\tau})$ preprocessing time
- A Monte-Carlo solution
  - $O\left(\tau \log\left(\mathrm{LCE}(i,j)/\tau\right)\right)$ query time (correct with high prob.)
  - $O((n+m)/\tau)$ space (even during preprocessing)
  - $O(n)$ preprocessing time.
- A Las-Vegas solution
  - $O\left(\tau \log\left(\mathrm{LCE}(i,j)/\tau\right)\right)$ query time (correct with certainty)
  - $O((n+m)/\tau)$ space (even during preprocessing)
  - $O(n \log n)$ preprocessing time with high prob.